2816. [2003 : 111] *Proposed by Boris Harizanov, student, Stara Zagora, Bulgaria.*

In acute-angled isosceles triangles $A_1B_1C_1$ (with $A_1C_1 = B_1C_1$) and $A_2B_2C_2$ (with $A_2C_2 = B_2C_2$), we have $A_1C_1 = A_2C_2$. For $k = 1, 2$, we have a circle with centre I_k and radius r_k inscribed in $\triangle A_kB_kC_k$, and a circle with centre O_k and radius R_k circumscribed around $\triangle A_kB_kC_k$.

If $I_1O_1 = I_2O_2$, is it true that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ must be congruent?

Solution by Christopher J. Bradley, Bristol, UK.

The answer is NO.

In a triangle with sides a, a and c, the circumradius R and the inradius r are given by $R = \dfrac{a^2}{\sqrt{4a^2 - c^2}}$ and $r = \dfrac{c}{2}\sqrt{\dfrac{2a - c}{2a + c}}$. Thus,

$$OI^2 \;=\; R^2 - 2Rr \;=\; \frac{a^2(a-c)^2}{4a^2 - c^2}\,.$$

If we have two triangles with sides a, a, c and a, a, d, then

$$\begin{aligned} O_1I_1^2 \;=\; O_2I_2^2 &\implies \frac{(a-c)^2}{4a^2 - c^2} \;=\; \frac{(a-d)^2}{4a^2 - d^2} \\ &\implies c = d \quad \text{or} \quad (2c - 5a)(2d - 5a) \;=\; 9a^2\,. \end{aligned}$$

Thus, the triangles are not necessarily congruent.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Some solvers did not mention the case $OI = 0$ (but this editor has been kind to them). Zhou commented that the answer is, in fact, "YES" if IO means the **signed distance**.

2817. [2003 : 112] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that A, B, and C are the angles of $\triangle ABC$. Define

$$\begin{aligned} L &= 4\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)\,; \\ M &= \left(\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right) \\ &\qquad \prod_{\text{cyclic}}\left(\cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) - \cos\left(\frac{A}{2}\right)\right)\,. \end{aligned}$$

Show that $L = M$.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Firstly,

$$\begin{aligned} M &= \left(\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)+2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right)\right) \\ &\quad\cdot\left(\cos^2\left(\frac{A}{2}\right)-\cos^2\left(\frac{B}{2}\right)-\cos^2\left(\frac{C}{2}\right)+2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\right) \\ &= 4\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right) \\ &\qquad - \left(\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right)\right)^2 . \qquad (1) \end{aligned}$$

Since $\cos\left(\frac{B+C}{2}\right)=\sin\left(\frac{A}{2}\right)$, we have

$$\begin{aligned} &\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right) \\ &\qquad = \tfrac{1}{2}(\cos B+\cos C+2)-\left(1-\sin^2\left(\frac{A}{2}\right)\right) \\ &\qquad = \cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)+\sin^2\left(\frac{A}{2}\right) \\ &\qquad = \sin\left(\frac{A}{2}\right)\left(\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{B+C}{2}\right)\right) \\ &\qquad = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) . \qquad (2) \end{aligned}$$

Substituting (2) into (1), we then have

$$M = 4\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)\left(1-\sin^2\left(\frac{A}{2}\right)\right) = L .$$

II. *Solution by Arkady Alt, San Jose, CA, USA.*

Let $\alpha=\frac{\pi-A}{2}$, $\beta=\frac{\pi-B}{2}$, and $\gamma=\frac{\pi-C}{2}$. Then α, β, $\gamma>0$, and $\alpha+\beta+\gamma=\pi$. Thus, we can interpret α, β, γ as the angles of a triangle T. Without loss of generality, we may assume that T has circumradius $R=1/2$.

Let a, b, and c denote the sides of T. Then, by the Law of Sines, we have $a=\sin\alpha$, $b=\sin\beta$, and $c=\sin\gamma$. Note that

$$\cos\frac{A}{2} = \sin\left(\frac{\pi}{2}-\frac{A}{2}\right) = \sin\alpha = a .$$

Similarly, $\cos\left(\frac{B}{2}\right) = b$ and $\cos\left(\frac{C}{2}\right) = c$. Then, using the well-known formula $R=\frac{abc}{4K}$, where K denotes the area of T, we have

$$L = 4(abc)^2 = 64R^2K^2 = 16K^2 .$$

On the other hand, using Heron's Formula, we get

$$\begin{aligned} M &= (\sin\alpha+\sin\beta+\sin\gamma)\prod_{\text{cyclic}}(\sin\alpha+\sin\beta-\sin\gamma) \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) = 16K^2 . \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; JOSEPH LING, University of Calgary, Calgary, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2818. [2003 : 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that n, $k \geq 2$ are integers such that $(n + k^n, k) = 1$.

Prove that at least one of $n + k^n$ and $n\,k^{(k^n - 1)} + 1$ is not prime.

Solution by Michel Bataille, Rouen, France.

Let $p = n + k^n$ and $q = nk^{(k^n-1)} + 1$. If p is composite, then we are done. Suppose instead that p is a prime. We are given that $(p, k) = 1$. By Fermat's Little Theorem, we have

$$q \;=\; (p - k^n)k^{p-n-1} + 1 \;\equiv\; -k^{p-1} + 1 \;\equiv\; 0 \pmod{p}\;.$$

Furthermore,

$$\begin{aligned} n\left(k^{(k^n-1)} - 1\right) \;&>\; k^{(k^n-1)} - 1 \;=\; \big(1 + (k-1)\big)^{(k^n-1)} - 1 \\ &\geq\; (k^n - 1)(k - 1) \;\geq\; k^n - 1\,, \end{aligned}$$

and hence, $q = nk^{(k^n-1)} + 1 > n + k^n = p$. It follows that q is composite.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Bergen, Norway; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Most of the solutions are the same as the one featured above, except for the demonstration of the fact that $p < q$, which almost all solvers either took for granted or simply stated as "evident" or "clear". From the proof given above, it is obvious that the assumption "$(n + k^n, k) = 1$" is really superfluous. However, Parmenter is the only solver who explicitly pointed this out.

2819. [2003 : 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy, for all real x and y, $f\left(\frac{2x+y}{3}\right) \geq f\left(\sqrt[3]{x^2y}\right)$.

Prove that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.